Group Based Interference Alignment

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Abstract—In the K-user single-input single-output (SISO) frequency-selective fading interference channel, it is shown that the maximal achievable multiplexing gain is almost surely $K/2$ by using interference alignment (IA). However, when the signaling dimensions are limited, allocating all the resources to all users simultaneously is not optimal. So, a group based interference alignment (GIA) scheme is proposed, and it is formulated as an unbounded knapsack problem. Optimal and greedy search algorithms are proposed to obtain group patterns. Analysis and numerical results show that the GIA scheme can obtain a higher multiplexing gain when the resources are limited.

Index Terms—Interference channel, interference alignment, multiplexing gain, knapsack problem.

I. INTRODUCTION

INTERFERENCE management is an important problem in wireless system design. As an effective technique for interference management, interference alignment (IA) is first considered in [1], [2] as a coding technique for the two-user multiple-input multiple-out (MIMO) X channel. Using this scheme, it is shown that each user can obtain almost surely a multiplexing gain (MG) of $1/2$ per channel use in the K-user SISO interference channel (IC) [3]. A beamforming matrices optimized IA (called BF-IA) scheme is proposed in [4] which can obtain a higher MG than the IA scheme in [3] at any given number of channel realizations. IA scheme is also applied in cellular networks in [5] which can boost system performance in some scenarios. However, when the signaling dimensions are limited, allocating all the resources to all users simultaneously is not optimal.

In this letter, a GIA scheme is proposed based on the BF-IA scheme which can obtain a higher MG when the resources are limited.

II. SYSTEM MODEL AND PRELIMINARIES

Consider the K-user frequency-selective fading IC model:

$$\mathbf{Y}^{[k]} = \sum_{l=1}^{K} \mathbf{H}^{[kl]}\mathbf{X}^{[l]} + \mathbf{Z}^{[k]}, \ \forall k \in \{1,\dots,K\}. \quad (1)$$

$\mathbf{X}^{[l]}$ is the $M \times 1$ input signal vector of the lth transmitter, and $\mathbf{Y}^{[k]}$ is the channel output at the kth receiver, where M is the number of available frequency-selective channel realizations. $\mathbf{H}^{[kl]}$ is the diagonal channel matrix between

transmitter l and receiver k. We assume that all $\mathbf{H}^{[kl]}$'s are known in advance at all the transmitters and all the receivers, and assume the channel is time-invariant. $\mathbf{Z}^{[k]}$ is $M \times 1$ additive white Gaussian noise (AWGN) vector at the kth receiver, where all noise terms are independent identically distributed (i.i.d.) zero-mean complex Gaussian with unit variance. The definition of achievable sum-rate follows from [3]. Define $r = \lim_{\text{SNR} \to \infty} \frac{R(\text{SNR})}{\log(\text{SNR})}$ as the MG [6], where $R(\text{SNR})$ is the achievable sum-rate in the K-user IC, and SNR is defined as the total transmit power across all transmitters. The frequency-selective channel realizations are called channel uses or resources for convenience in this letter.

A. Original Interference Alignment (OIA) in the K-User IC

Let $N = (K-1)(K-2) - 1$, $M = (n+1)^N + n^N$ (n is a positive integer), and let $(d^{[1]}, d^{[2]}, \dots, d^{[K]}) = ((n+1)^N, n^N, \dots, n^N)$ be the numbers of streams allocated to the K users respectively. Then

$$\{r^{[1]}, r^{[2]}, \dots, r^{[K]}\} = \left\{ \frac{(n+1)^N}{(n+1)^N + n^N}, \right.$$
$$\left. \frac{n^N}{(n+1)^N + n^N}, \dots, \frac{n^N}{(n+1)^N + n^N} \right\} \quad (2)$$

are the achievable MGs of the K-user IC over M channel uses. So, the total achievable MG over M channel uses is [3]

$$r_{OIA}(K, M) = \sum_{i=1}^{K} r^{[i]} = \frac{(n+1)^N + (K-1)n^N}{(n+1)^N + n^N}. \quad (3)$$

B. Beamforming Optimized Interference Alignment (BF-IA)

An efficient IA scheme is proposed in [4], where the precoding matrices are optimized. The streams allocated to the users are

$$\{d^{[1]}, d^{[2]}, \dots, d^{[K]}\} =$$
$$\left\{ \binom{n^* + N + 1}{N}, \binom{n^* + N}{N}, \dots, \binom{n^* + N}{N} \right\}, \quad (4)$$

and the dimension of the extended channel is

$$M = d^{[1]} + d^{[2]} = \binom{n^* + N + 1}{N} + \binom{n^* + N}{N}, \quad (5)$$

where $N = (K-1)(K-2) - 1$ and n^* is a nonnegative integer. The total achievable MG is [4]

$$r_{BF}(K, M) = \frac{d^{[1]} + (K-1)d^{[2]}}{d^{[1]} + d^{[2]}} \quad (6)$$

$$= \frac{(K-1)(n^* + 1) + n^* + N + 1}{2n^* + N + 2}. \quad (7)$$

For example, when $K = 4$, $N = (4-1)(4-2) - 1 = 5$, a solution to IA is feasible over the following dimensions of the

extended channel: $\mathcal{L}_4 = \{7, 27, 77, 182, \dots\}$. $\mathcal{L}_k$ is defined as the set of all the feasible length of the extended channel over k users when $k \geq 3$. When $k < 3$, orthogonal multiplexing is MG optimal. So, let $\mathcal{L}_1 = \mathcal{L}_2 = \{1, 2, 3, \dots\}$.

When $M \to \infty$, we have

$$\lim_{M \to \infty} r_{BF}(K, M) = \lim_{n^* \to \infty} \frac{Kn^* + K + N}{2n^* + N + 2} = \frac{K}{2}. \quad (8)$$

Hence, allocating all the resources to all users simultaneously is MG optimal.

Using (6), Fig. 1 illustrates the achievable MG when $K = 3$ and $K = 4$. It can be seen that when $M \geq 77$ allocating all the resources to all users simultaneously ($K = 4$) can obtain more MG, and when $M < 77$ allocating them to partial users ($K = 3$) can obtain more MG. For example, when $M = 7$, $r_{BF}(4, 7) \approx 1.2857$ while $r_{BF}(3, 7) \approx 1.4286$.

For general values of M and K, a natural question is how to allocate resources among users can obtain more MG. In the following section a GIA scheme is proposed based on the BF-IA scheme.

III. GROUP BASED INTERFERENCE ALIGNMENT

Let M be the total resources that can be used for IA in the K-user IC. Let $k \leq K$, $m \in \mathcal{L}_k$ and $m \leq M$, and let

$$v = \begin{cases} \frac{m}{M}[r_{BF}(k, m) - 1] & k \geq 3 \\ 0 & k < 3 \end{cases} \quad (9)$$

be the relative MG that the BF-IA scheme obtained compared to orthogonal multiplexing scheme (when $k < 3$, orthogonal multiplexing is MG optimal). Define $e = \{k, m, v\}$ as a group pattern. For example, in Fig. 2a we have $e = \{7, 45880, 0.5405\}$ while in Fig. 2b we have $e_1 = \{4, 35853, 0.6759\}$, $e_2 = e_3 = \{4, 5005, 0.0873\}$, and $e_4 = \{3, 17, 0.0002\}$.

Let $\mathcal{S}_k^M = \{\{k, m_1, v_1\}, \{k, m_2, v_2\}, \dots, \{k, m_w, v_w\}\}$ be the set of all the group patterns over k users exactly, where $m_j \leq M$, $1 \leq j \leq w$. Define the relative MG obtained per dimension as

$$\rho_j = \begin{cases} \frac{v_j}{m_j} = \frac{1}{M}[r_{BF}(k, m_j) - 1] & k \geq 3 \\ 0 & k < 3 \end{cases} \quad (10)$$

which is the *efficiency* of a group pattern. For example, in Fig. 2a we have $\rho \approx 1.1782 \times 10^{-5}$ while in Fig. 2b we have $\rho_1 \approx 1.8851 \times 10^{-5}$, $\rho_2 = \rho_3 \approx 1.7437 \times 10^{-5}$, and $\rho_4 \approx 1.0257 \times 10^{-5}$.

Let $\mathcal{E}_K^M = \bigcup_{j=1}^{K} \mathcal{S}_j^M$ be the set of all the group patterns over any k users when $k \leq K$. We denote the elements of e_i as $e_i.k$, $e_i.m$, and $e_i.v$ respectively. If there exist two group patterns e_i and e_l with $e_i.m \leq e_l.m$ and $e_i.v \geq e_l.v$ in the set $\mathcal{E}_K^M$, then it would always be better (or at least not worse) to choose e_i. Hence, e_l is removed from the set $\mathcal{E}_K^M$, and we obtain $\overline{\mathcal{E}}_K^M = \{e_1, \dots, e_W\}$, $W = |\overline{\mathcal{E}}_K^M|$. For example, if $e_i = \{3, 7, 6.5388 \times 10^{-5}\} \in \mathcal{S}_3^{45880}$ and $e_l = \{4, 7, 4.3592 \times 10^{-5}\} \in \mathcal{S}_4^{45880}$, then e_l is removed from the set $\mathcal{E}_K^M$.

The elements of $\overline{\mathcal{E}}_K^M$ are sorted by ρ in non-increasing order, and we obtain $\overleftarrow{\mathcal{E}}_K^M = \{e_1, \dots, e_W\}$. For example, when $K = 7$ and $M = 45880$, we have $\overleftarrow{\mathcal{E}}_K^M =$
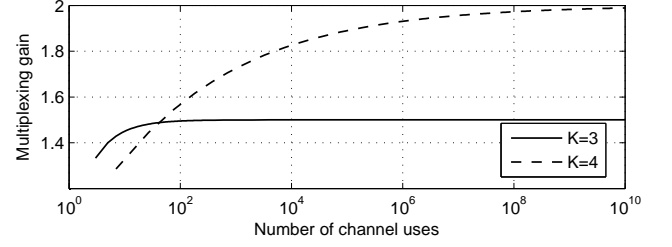

Fig. 1. The achievable MG of the BF-IA scheme when $K = 3$ and $K = 4$.

$\{\{4, 35853, 0.6759\}, \{4, 27132, 0.5069\}, \{4, 20196, 0.3735\}, \{5, 44200, 0.8092\}, \dots\}$ where $\rho_1 \approx 1.8851 \times 10^{-5}$, $\rho_2 \approx 1.8682 \times 10^{-5}$, $\rho_3 \approx 1.8494 \times 10^{-5}$, $\rho_4 \approx 1.8309 \times 10^{-5}$, $\dots$, and $W = 470$.

Given K and M, we have obtained all the group patterns sorted by efficiency, and the question becomes how to choose among them so as to obtain more MG. It is modeled as an unbounded knapsack problem which is NP-hard [7]. The corresponding integer programming formulation is given as follows.

$$\max \sum_{j=1}^{W} x_j \cdot e_j.v \quad (11)$$

$$\text{subject to } \sum_{j=1}^{W} x_j \cdot e_j.m \leq M \quad (12)$$

$$x_j \geq 0, \text{integer}, j = 1, \dots, W. \quad (13)$$

We denote the solution values by z_o (optimal algorithm) and z_g (greedy algorithm), and denote the solution sets by $\mathcal{P}_o$ (optimal algorithm) and $\mathcal{P}_g$ (greedy algorithm) respectively.

A. Optimal Search Algorithm

As a standard dynamic programming algorithm, Unbounded-DP [7] is adapted to evaluate our problem.

Algorithm optimal
  for $m := 0$ to M do
    $z(m) := 0$, $r(m) := 0$ % *initialization*
  for $j := 1$ to W do
    for $m := e_j.m$ to M do   % e_j *may be packed*
      if $z(m - e_j.m) + e_j.v \geq z(m)$ then
        $z(m) := z(m - e_j.m) + e_j.v$
        $r(m) := j$
  $\mathcal{P} := \varnothing$, $\overline{m} := M$
  repeat   % *recover the optimal solution set*
    $r := r(\overline{m})$
    $\mathcal{P} := \mathcal{P} \bigcup \{e_r\}$
    $\overline{m} := \overline{m} - e_r.m$
  until $\overline{m} = 0$
  $z_o := z(M)$, $\mathcal{P}_o := \mathcal{P}$

Let $K = 7$, $M = 45880$ ($\in \mathcal{L}_7$). Fig. 2b shows the search results of the optimal algorithm. The total MG obtained is $r_G^{opt} = z_o + 1 \approx 1.8506$. Using (6), the MG obtained by the BF-IA scheme is about 1.5405. So, about 20% more MG is obtained by making full use of 45880 channel uses.

The computation complexity of this optimal algorithm is $O(MW)$, and it is a pseudopolynomial algorithm [7].

B. Greedy Search Algorithm

When M is large, the optimal algorithm will need prohibitive time to obtain the solutions even using a powerful computer. So, a greedy algorithm is proposed in the following:

Algorithm greedy
$\quad m := 0,\ z := 0,\ \mathcal{P} := \varnothing$
$\quad$ for $j := 1$ to W do
$\quad\quad$ if $m + e_j.m \le M$ then
$\quad\quad\quad x_j := \lfloor (M - m)/e_j.m \rfloor$
$\quad\quad\quad m := m + x_j \cdot e_j.m$
$\quad\quad\quad z := z + x_j \cdot e_j.v$
$\quad\quad\quad \mathcal{P} := \mathcal{P} \bigcup \underbrace{\{e_j, \ldots, e_j\}}_{x_j}$
$\quad z_g = z,\ \mathcal{P}_g = \mathcal{P}$

The computation complexity of the greedy algorithm is $O(W)$. However, as the greedy algorithm does best every step which is locally optimal, global optimum is not confirmed. According to *Theorem* 8.5.1 in [7], the relative performance guarantee (z_g/z_o) of the greedy algorithm is bounded by $1/2$. Fig. 2c shows the search results of the greedy algorithm when $K = 7$ and $M = 45880$. In this example, the total MG obtained is $r_G^{grd} = z_g + 1 \approx 1.8494$, and the relative performance guarantee is about 99.86%.

C. Discussions

When each user is equipped with T antennas, let $K' = KT$, $N' = (K'-1)(K'-2) - 1$, and $M' = \binom{n^*+N'+1}{N'} + \binom{n^*+N'}{N'}$. Using *Corollary* 1 in [4], the achievable MG is $r_{BF}(K', M')$. So, the GIA scheme and the search algorithms can be readily extended to this scenario.

The stream allocations among users are non-uniform both in the BF-IA and the GIA schemes. Hence, dynamic algorithm should be considered to balance unfairness among users. There is a simple solution, that is choosing the user served with more streams from all users periodically. Also, sophisticated schemes can be designed to incorporate other factors. However, we leave it for future work.

IV. NUMERICAL RESULTS

The comparison of the GIA and the BF-IA schemes in the K-user IC is presented in Fig. 3. We only compare the achievable MG when $m \in \mathcal{L}_7$ in Fig. 3a (or $m \in \mathcal{L}_{14}$ in Fig. 3b), as the BF-IA scheme has no solution when $m \notin \mathcal{L}_7$ (or $m \notin \mathcal{L}_{14}$). And we also only compare the greedy algorithm of the GIA scheme with the BF-IA scheme, as the optimal algorithm is only feasible when M is small.

Fig. 3 shows that the proposed GIA scheme obtains a higher MG when the resources below a certain value (e.g., about 10^{17} in Fig. 3a). When the resources are large, as discussed above, allocating all them to all users simultaneously is MG optimal. The unsteadiness of the curves of the GIA scheme is because the discrete nature of the problem. The GIA scheme is preferred to be used when the resources are limited.
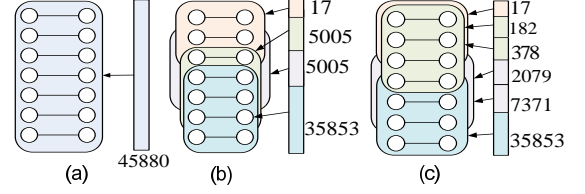

Fig. 2. Comparison of the BF-IA and the GIA schemes when $K = 7$ and $M = 45880$. (a) The BF-IA scheme, which allocates all the resources to 7 users simultenously. (b) The GIA scheme (optimal algorithm), where M is divided into 4 parts ($m_1 = 35853$, $m_2 = m_3 = 5005$, and $m_4 = 17$). (c) The GIA scheme (greedy algorithm), where M is divided into 6 parts ($m_1 = 35853$, $m_2 = 7371$, $m_3 = 2079$, $m_4 = 378$, $m_5 = 182$, and $m_6 = 17$).
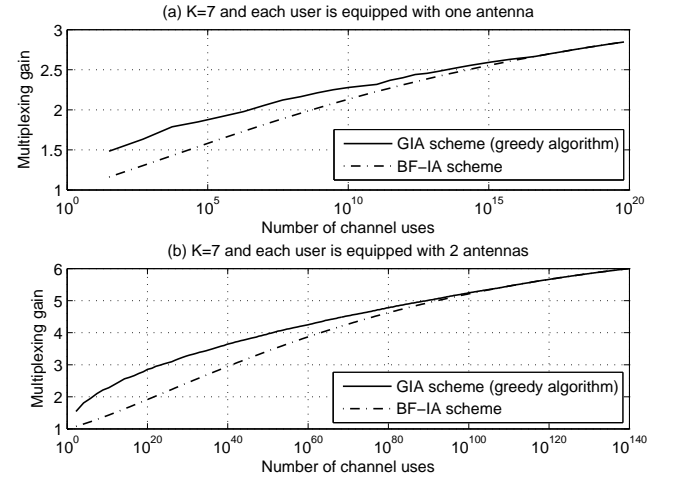

Fig. 3. Comparison of the GIA (greedy algorithm) and the BF-IA schemes in the K-user IC. (a) $K = 7$ and each user is equipped with one antenna. (b) $K = 7$ and each user is equipped with 2 antennas, i.e., $K' = KT = 14$.

V. CONCLUSION

In this letter, under the same IA conditions as in [4], a group based IA scheme is proposed. Analysis and numerical results show that the GIA scheme obtains a higher MG in comparison with the BF-IA scheme when the resources are limited.

REFERENCES

[1] M. Maddah-Ali, A. Motahari, and A. Khandani, "Signaling over MIMO multi-base systems: Combination of multi-access and broadcast schemes," *in Proc. IEEE Int. Symp.Inform.Theory*, 2006.
[2] M. Maddah-Ali, A. Motahari, and A. Khandani, "Communication over MIMO X channel: Interference alignment, decomposition, and performance analysis," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3457-3470, Aug. 2008.
[3] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom for the K user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425-3441, Aug. 2008.
[4] S. W. Choi, S.A. Jafar, and S.Y. Chung, "On the beamforming design for efficient interference alignment," *IEEE Commun. Lett.*, vol. 13, no. 11, pp. 847-849, Nov. 2009.
[5] C. Suh, M. Ho, and D. Tse, "Downlink interference alignment," in arXiv:cs.IT/1003.3707v2, May. 2010.
[6] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49. no. 5, pp. 1073-1096, May. 2003.
[7] H. Kellerer, U. Pferschy, and D. Pisinger, *Knapsack Problems*, Springer, 2004.